ELLIPTIC OPERATORS ON PLANAR GRAPHS: UNIQUE CONTINUATION FOR EIGENFUNCTIONS AND NONPOSITIVE CURVATURE

STEFFEN KLASSERT, DANIEL LENZ,
NORBERT PEYERIMHOFF AND PETER STOLLMANN

[1] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: S.Klassert@mathematik.tu-chemnitz.de,

[2] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: D.Lenz@mathematik.tu-chemnitz.de,

[3] Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany; E-mail: peyerim@math.ruhr-uni-bochum.de,

[4] Fakultät für Mathematik, Technische Universität Chemnitz, D-09107 Chemnitz, Germany; E-mail: P.Stollmann@mathematik.tu-chemnitz.de

Abstract. This paper is concerned with elliptic operators on plane tessellations. We show that such an operator does not admit a compactly supported eigenfunction, if the combinatorial curvature of the tessellation is nonpositive. Furthermore, we show that the only geometrically finite, repetitive plane tessellations with nonpositive curvature are the regular $(3,6),(4,4)$ and $(6,3)$ tilings.

1. Introduction

Since the work of Aronszajn [Aro], unique continuation for elliptic operators on Riemannian manifolds and Schrödinger operators has been investigated in very many papers of which we only mention [Ag, ABG, H, JK]. It was shown under rather general assumptions that a function f on a connected Riemannian manifold vanishes identically, whenever it vanishes in one point to infinite order and satisfies

$$Hf = 0 \tag{1}$$

where H is an elliptic operator. In particular, f satisfying (1) must vanish identically if it vanishes on the non empty complement of a a compact set.

Despite all analogies between elliptic operators on graphs and those on manifolds, unique continuation does not hold on graphs. It fact, it is rather easy to give examples of elliptic operators on graphs with compactly supported eigenfunctions. These examples have recently attracted some attention since they play a role in the investigation of the so called integrated density of states for random operators [DLMSchY, KLS, Ves].

More precisely, in [DLMSchY], Dodziuk et al. study a certain periodic Laplacian on graphs viz Laplacian on infinite graphs, which are coverings of finite graphs by amenable groups. They show that the eigenvalues of the whole graph operator are

the union of the eigenvalues of suitable restrictions to finite graphs. Moreover, they obtain a characterization of the points of discontinuity of the integrated density of states by existence of compactly supported eigenfunctions.

Independently, existence of compactly supported eigenfunctions was studied by three of the authors in [KLS] for certain aperiodically ordered graphs. This study does not only give examples of compactly supported eigenfunctions, but again links their occurrence to discontinuities of the integrated density of states.

More recently, there is related work of Veselić on high random graphs [Ves].

While a common framework to operators on these three classes is still missing, the link between discontinuities of the integrated density of states and the occurrence of compactly supported eigenfunctions is by now well established.

The aim of this paper is to investigate combinatorial conditions on the graph which guarantee nonexistence of compactly supported eigenfunctions.

In this context, the only result available so far is due to Delyon/Souillard [DS]. They show absence of compactly supported eigenfunctions for random Schrödinger operators on the d-dimensional lattice and use this to conclude continuity of the corresponding integrated density of states.

Here, we will restrict our attention to plane tessellating graphs and establish a connection between absence of compactly supported eigenfunctions of an elliptic operator and the combinatorial curvature of the graph introduced in [BP1]. Our main result uses the geometric/combinatorial methods developped in [BP1, BP2] and states the following:

Result 1 *If the curvature of the plane tessellation $\mathcal{G}$ is nonpositive then no elliptic operator on $\mathcal{G}$ admits a compactly supported eigenfunction.*

Note that our result implies Delyon/Souillard's result in the particular case of a two-dimensional lattice.

While the result apriori applies to general plane tessellating graphs, it is of limited use when it comes to application to geometrically more rigid tessellations with certain repetitivity properties (as are the ones encountered in [KLS]). Namely, our second result shows:

Result 2 *If a geometrically finite, repetitive plane tessellation has nonpositive curvature then it coincides with one of the three regular combinatorial tessellations* $(3,6)$*,* $(4,4)$ *or* $(6,3)$*.*

2. Notation and results

In this section we introduce the model which we consider and present our results.

A planar graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ is a graph which is embedded in the plane $\mathbb{R}^2$. Here, $\mathcal{V}$ denotes the set of vertices and $\mathcal{E}$ the set of edges of $\mathcal{G}$. $\mathcal{F}$ denotes the family of connected components of the complement of the image of $\mathcal{G}$ in $\mathbb{R}^2$. The elements of $\mathcal{F}$ are called *faces* of $\mathcal{G}$. We will always assume that our graphs are compactly finite, i.e. every point in $\mathbb{R}^2$ has a neighbourhood which meets only finitely many faces.

We call a face $f \in \mathcal{F}$ a *polygon* if $\bar{f}$ is homeomorphic to a closed disc and its boundary defines a simply closed path of vertices. In this case, the boundary is called the *boundary cycle* ∂f of f. The edges which belong to ∂f are called the *sides* of f. The number of edges of f is denoted by $E_{\partial f}$. If the polygon f has k sides it is called a *k-gon*. The number of edges emanating from a vertex $v \in \mathcal{V}$ is called the *degree of* v, denoted by $|v|$. Two vertices are called *adjacent* if they are connected by an edge. We write $v \sim w$ if v and w are adjacent vertices.

Definition 1. *A planar graph $\mathcal{G}$ in $\mathbb{R}^2$ is called* tessellating*, if the following conditions are satisfied:*

i) *Any edge is a side of precisely two different faces.*
ii) *Any two faces are disjoint or have precisely either a vertex or a side in common.*
iii) *Any face $f \in \mathcal{F}$ is a polygon with finitely many sides.*
iv) *Every vertex has finite degree.*

In order to present our results we have to introduce the corresponding notions of combinatorial curvature and of elliptic operators.

A *corner* of a tessellating graph is a pair $(v, f) \in \mathcal{V} \times \mathcal{F}$ so that $v \in \partial f$. The set of all corners of $\mathcal{G}$ is denoted by $\mathcal{C} := \mathcal{C}(\mathcal{G})$.

Definition 2. *Let $\mathcal{G}$ be a plane tessellation. Then, the function $\kappa : \mathcal{C} \to \mathbb{R}$ defined by*

$$\kappa(v, f) := \frac{1}{|v|} + \frac{1}{E_{\partial f}} - \frac{1}{2}$$

is called the the curvature *(on the graph $\mathcal{G}$). The graph $\mathcal{G}$ is said to have nonpositive curvature if $\kappa(v, f) \leq 0$ for every $(v, f) \in \mathcal{C}$.*

Let $C(\mathcal{V})$ be the vector space of all complex valued functions on $\mathcal{V}$. Let $C_c(\mathcal{V})$ be the subspace of $C(\mathcal{V})$ consisting of functions which vanish outside a finite set of vertices.

A linear operator $L : C(\mathcal{V}) \longrightarrow C(\mathcal{V})$ is called *elliptic* if its matrix $a : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ satisfies $a(v, w) \neq 0$ whenever $v \sim w$ and $a(v, w) = 0$ whenever $v \not\sim w$ and $v \neq w$. Thus

$$Lu(v) := \sum_{w \in \mathcal{V}} a(v, w)u(w) = a(v, v)u(v) + \sum_{w \sim v} a(v, w)u(w)$$

for every vertex v in $\mathcal{G}$. A well studied example is the nearest neighbour Laplacian where $a(v, v) = |v|$ for all $v \in \mathcal{V}$ and $a(v, w) = 1$ whenever $v \sim w$.

As usual, a non vanishing function $u \in C(\mathcal{V})$ is called an eigenfunction of the elliptic operator L to the eigenvalue λ if $Lu - \lambda u = 0$, i.e. if

$$\sum_{w \sim v} a(v, w)u(w) = (\lambda - a(v, v))u(v) \quad \text{for all } v \in \mathcal{V}.$$

The precise version of our main result now reads as follows:

Theorem 3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature and L an elliptic operator on $C(\mathcal{V})$. Then L does not admit an eigenfunction in $C_c(\mathcal{V})$.*

Note that the above theorem takes only the combinatorial structure of the tessellation into account and that the precise geometric shape of the faces is of no importance. Our next result, however, is concerned with geometrically more rigid tessellations $\mathcal{G}$. We assume that $\mathcal{G}$ is build up by copies of finitely many fixed geometric tiles $f_1, \ldots, f_N \in \mathcal{F}$ and that every finite configuration of tiles in $\mathcal{G}$ can be found repeatedly in any sufficiently large Euclidean ball.

Definition 4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation. Then $\mathcal{G}$ is called* geometrically finite, *if there are finitely many faces $\{f_1, \ldots, f_N\} \subset \mathcal{F}$, the* generators *of $\mathcal{G}$, such that every $f \in \mathcal{F}$ is an isometric image of one of the generators, i.e., $f = t + Af_j$, for a $j \in \{1, \ldots, N\}$, $t \in \mathbb{R}^2$ and $A \in SO(2)$. A geometrically finite tessellation $\mathcal{G}$ is called* repetitive, *if for every finite set of faces $\{g_1, g_2, \ldots, g_k\} \subset \mathcal{F}$ there is an $R > 0$ with the following property: In any Euclidean ball $B_R(x) \subset \mathbb{R}^2$ there are k faces $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_k \in \mathcal{F}$ and $t \in \mathbb{R}$, $A \in SO(2)$, such that $\hat{g}_j = t + Ag_j$, for $j = 1, \ldots, k$.*

Theorem 5. *A geometrically finite, repetitive plane tessellation $\mathcal{G}$ of* nonpositive curvature *coincides with one of the three regular combinatorial tessellations* $(3,6)$*,* $(4,4)$ *or* $(6,3)$.

The three regular tessellations are illustrated in [GS, Figure 1.2.1].

3. Geometry of plane tessellations

In this section we discuss some geometric aspects of plane tessellations following [BP1, BP2]. Moreover, for non positively curved plane tessellations we prove the impossibility of a certain vertex labeling of the boundary cycle of distance balls.

Let a plane tessellation $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be given.

Two faces in $\mathcal{F}$ are called *neighbours*, if they have an edge in common. A sequence $(f_0, f_1, \dots,)$ of faces is a *(connected) path*, if any two subsequent faces f_j, f_{j+1} are neighbours. If this sequence is finite i.e. of the form $(f_0, f_1, \dots, f_n)$, the length of this path is defined to be n. The *(combinatorial) distance* $d(f,g)$ of two faces f and g is the smallest number n for which there exists a path $(f_0, f_1, \dots, f_n)$ with $f_0 = f$ and $f_n = g$.

Similarly, for a finite set P in $\mathcal{F}$ and $f \in \mathcal{F}$ arbitrary, we define $d(f,P) := \min\{d(f,g) : g \in P\}$. Thus, for each finite set P and any $k \in \mathbb{N}_0$ we can define the k-neigbourhoods of P by

$$B_k(P) := \{f \in \mathcal{F} \mid d(f,P) \leq k\}.$$

A special example of much relevance in our considerations arises if P consists of only one element: For a given face f_0 and $k \in \mathbb{N}_0$, we define the *distance balls*

$$B_k(f_0) = \{f \in \mathcal{F} \mid d(f,f_0) \leq k\}. \tag{2}$$

Accordingly, we define the *distance spheres* by

$$A_k(f_0) = \{f \in \mathcal{F} \mid d(f,f_0) = k\}. \tag{3}$$

Apparently, each distance ball arises from the previous distance ball by adding a distance sphere.

It turns out that this gives a very well behaved "onion-like" layered structure on the distance balls, provided $\mathcal{G}$ has nonpositive curvature. This layered structure is the crucial ingredient in the proofs of our main results. It was analyzed in detail in the works [BP1, BP2].

In order to discuss the relevant results we need some more definitions.

A finite subset P of $\mathcal{F}$ is called a *polygon*, if $\bigcup_{f \in P} \overline{f} \subset \mathbb{R}^2$ is homeomorphic to a closed disc. Then, ∂P denotes the cycle of boundary vertices. If P is a polygon and v belongs to ∂P, we define the *inner degree of* V *with respect to* P by

$$|v|_P^i := \text{ number of faces of } f \in P \text{ meeting in } v$$

and the *exterior degree of* v *with respect to* P by

$$|v|_P^e := |v| - |v|_P^i.$$

We introduce now a labeling of the boundary vertices of the polygon P with the letters a and b. First, for any vertex v of $\mathcal{G}$, we define

$$N(v) = \min\{E_{\partial f} \mid f \in \mathcal{F} : v \in \partial f\}.$$

A vertex $v \in \partial P$ obtains label a if $|v|_P^i = 1$, or if both $N(v) = 3$ and $|v|_P^i \leq 3$ hold. All other vertices of ∂P obtain label b. Moreover, if v is an a-vertex satisfying $|v|_P^i = 1$ then we say that v is of type a^+. One should think of the labeling as a sort of book-keeping of convexity: a^+-vertices of ∂P are considered to be particularly convex and b-vertices to be particularly concave.

The following proposition gives a simple fact on this labeling.

Proposition 6. *Let $\mathcal{G}$ be a plane tessellation with nonpositive curvature. Let P be a polygon. If $v \in \partial P$ satisfies $|v|_P^e = 1$, then no edge starting in v belongs to $\partial B_1(P)$. Moreover, in this case v has a b-label.*

Proof. The first statement is obvious from the definition of the exterior degree. If v would be an a-vertex then it would satisfy $|v| \leq 4$ and it would be adjacent to a triangle, in contradiction to nonpositive curvature. □

We say $w = (v_0, v_1, v_2 \dots, v_k)$ is a connected (vertex-)path of length $|w| = k$ in $\mathcal{G}$ if all subsequent vertices of w are connected by an edge. A connected path $w \subset \partial P$ is called *admissible* if for every vertex $v \in w$ with label b its neighbours in w carry the label a^+. We call the polygon P *admissible* iff ∂P is admissible. Thus, admissible polygons have the property that particularly concave vertices are compensated by particularly convex vertices in their neighbourhood.

It turns out that admissibility is preserved under taking k-neigbourhoods if the curvature in nonpositive. More precisely, the following is proven in Proposition 2.5 and Proposition 2.6 of [BP2]:

Lemma 7. *Let $\mathcal{G}$ be a plane tessellation with nonpositive curvature. Let P be an admissible polygon. Then the set $B_1(P)$ is an admissible polygon. Moreover, for every face $f \in B_1(P) - P$,*

(a) *$\partial f \cap \partial P$ is a connected path of edges of length ≤ 2 and*
(b) *$\partial f \cap \partial B_1(P)$ is a connected path of length ≥ 1.*

Apparently, every face $f \in \mathcal{F}$ is admissible. Thus, the lemma immediately implies (see Corollary 2.7 in [BP2] as well):

Lemma 8. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature. Let $f \in \mathcal{F}$ be arbitrarily. Then, every distance ball $B_k(f)$ is an admissible polygon and every face of the distance sphere $A_k(f)$ contributes at least one edge to the boundary $\partial B_k(f)$.*

Now, the main result of [BP2] is the following combinatorial analogue of the Hadamard-Cartan theorem in differential geometry. Note, that it can be seen as describing a nice layered structure of $\mathcal{G}$.

Theorem 9. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation. For a given face $f \in \mathcal{F}$, we define the cut locus $C(f) \subset \mathcal{F}$ of f in the metric space $(\mathcal{F}, d)$ to be the set*

$$C(f) := \{g \in \mathcal{F} \mid d(f', f) \leq d(g, f) \text{ for all neighbours } f' \text{ of } g\},$$

i.e., the set of all faces on which the distance function $d_f(g) = d(f, g)$ attains a local maximum. If $\mathcal{G}$ has nonpositive curvature, then $C(f) = \emptyset$ for all $f \in \mathcal{F}$.

This theorem has the following consequence:

Lemma 10. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature. Let $f \in \mathcal{F}$ be arbitrarily. Then the boundary $\partial B_k(f)$ imposes a cyclic enumeration $(f_1, \dots, f_n)$ of $A_{k+1}(f)$ such that precisely subsequent faces intersect and each intersection contains a vertex $v \in \partial B_k(f)$.*

Proof. This follows from Theorem 3.2 of [BP1] and the fact that $\mathcal{G}$ has no cut locus by the previous theorem. □

We are heading towards the result on the vertex labeling mentioned in the beginning of this section.

We will use the following lemma from [BP2]. The lemma is rather technical but very important for the proof of Theorem 3.

Lemma 11 (Lemma 2.8 in [BP2]). *Let $\mathcal{G}$ be a plane tessellation with nonpositive curvature and $A_k(f) = \{f_1, f_2, \dots, f_n\}$ be a distance sphere in $\mathcal{G}$ with cyclic enumeration of its faces. Then there is at least one face $f_j \in A_k(f)$ with one of the following properties: either $|\partial f_j \cap \partial B_{k-1}(f)| = 1$ or f_j does not share a common edge with both f_{j-1} and f_{j+1} (mod n).*

The following proposition is a key step in our proof. It may be of independent interest.

Proposition 12. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature and B_k be a distance ball for a given $f_0 \in \mathcal{F}$ with the closed simple sequence of vertices $v_0, v_1, v_2 \dots, v_{l-1}, v_l = v_0$ describing ∂B_k.*

If l is even and every b-vertex on ∂B_k satisfies $|v|^e_{B_k} = 1$, then the closed simple label-sequence of vertices describing ∂B_k is not of the form a^+, b, a^+, b, a^+, b, …, a^+, b.

Proof. Note that $\partial B_0 = \partial f_0$ is of the form $a^+, a^+, \dots, a^+$. Let $k \geq 1$ and assume that the label sequence of B_k is of the form $a^+, b, \dots, a^+, b$. Then, for every $f \in A_k := A_k(f_0)$, $\partial f \cap \partial B_k$ is a connected path of length ≤ 2 (for otherwise there were at least two successive vertices of label a^+). By Lemma 10, the faces of A_k can be enumerated consecutively such that only faces with subsequent indices intersect. We call faces in A_k with such a nonempty intersection "neighbours in A_k".

Case 1: There is a b-vertex v of ∂B_k satisfying $|v|^i_{B_k} = 2$.

As $|v|^e_{B_k} = 1$, we have $|v| = 3$. Thus, non positivity of the curvature implies that all faces adjacent to v have to be at least 6-gons. Let $f \in B_k$ be a face with $v \in \overline{f}$. Note that f must belong to A_k, as $|v| = 3$. Therefore, the edges of f fall into three different types: those which belong to ∂B_{k-1}, those which belong to ∂B_k and those which are a common edge with one of the two neighbours of f in A_k.

As all distance balls are admissible, Lemma 7 (a) gives $|\partial f \cap \partial B_{k-1}| \leq 2$. Together with the fact $|\partial f \cap \partial B_k| \leq 2$ we conclude that f has to be a 6-gon, sharing an edge with both its neighbours. Moreover, both b-vertices of ∂B_k belonging to $\overline{f}$ satisfy $|v|^i_{B_k} = 2$.

By repetition of the above arguments this implies that, consecutively, every face of A_k has to be a 6-gon, sharing an edge with both its neighbours. This situation is not possible, by Lemma 11.

Case 2: Every b-vertex of ∂B_k satisfies $|v|^i_{B_k} \geq 3$.

This means that none of the faces f in A_k shares a common edge with any of its neighbours in A_k, and since $\partial f \cap \partial B_{k-1}$ is a connected path of at most 2 edges (see Lemma 7 (a)), all faces $f \in A_k$ are at most 4-gons.

Case 2.1: A_k contains a 4-gon f.

The face f contributes an edge to the boundary of B_k and therefore at least one of its edges carries an a^+ label. Thus, f contributes precisely two edges to the boundary of B_k and there exists a unique vertex v of ∂f which does not belong to ∂B_k.

Assume that v is an a-vertex with respect to the labeling of ∂B_{k-1}. Then v is adjacent to a triangle and we have $|v|^i_{B_{k-1}} \geq 5$, for curvature reasons, which contradicts to label a.

Hence, v is an b-vertex with respect to B_{k-1}. The neighbours of v along ∂B_{k-1} are then a^+-vertices (since B_{k-1} is admissible). Therefore, the two b-vertices v', v'' with respect to B_k which belong to f are also a^+-vertices with respect to B_{k-1}, i.e., we have $|v'|^i_{B_k} = 3$. Thus we have $|v'| = 4$ because of $|v'|^e_{B_k} = 1$, and the neighbour f' of f in A_k with $v' \in \partial f'$ cannot be a triangle, for curvature reasons.

This shows that all faces in A_k are 4-gons and that we have, again, for B_{k-1} the situation that the vertices of ∂B_{k-1} are labeled as $a^+, b, a^+, b, \dots, a^+, b$ and

all b-vertices satisfy $|v|^e_{B_{k-1}} = 1$ and $|v|^i_{B_{k-1}} \geq 4$, for curvature reasons. Again, by curvature reasons, A_{k-1} must contain a 4-gon, since there are vertices $v' \in \partial B_k \cap \partial B_{k-1}$ with $|v'| = 4$. Thus, we may apply induction, and conclude that $B_0 = \{f_0\}$ is a 4-gon with label-sequence a^+, b, a^+, b, which is a contradiction.

Case 2.2: A_k consists only of triangles.

The label sequence forces each triangle to contribute two edges to the boundary of B_k. Let f be a triangle in A_k. For curvature reasons, the two b-vertices (with respect to B_k) of f satisfy $|v|^i_{B_{k-1}} \geq 3$ (note that $|v|^e_{B_k} = 1$). This means that the face f' of B_{k-1} which shares an edge with f does not share a common edge with both of its neighbours in A_{k-1} and that $|\partial f \cap \partial B_{k-1}| = 1$. Since $\partial f' \cap \partial B_{k-2}$ is a connected path of at most 2 edges, by Lemma 7 (a), f' has to be, again, a triangle.

Note that, by curvature reasons, the unique vertex of f' which is not also a vertex of f has to be a b-vertex with respect to B_{k-2}. Moreover, by admissibility of B_{k-2}, its label sequence is, again, given by $a^+, b,\ a^+, b, \ldots, a^+, b$, and that all b-vertices of ∂B_{k-2} satisfy $|v|^e_{B_{k-2}} = 0$. Now, B_{k-2} satisfies, again, the conditions of the proposition. Since neither Case 1 nor Case 2.1 can be given, we conclude that, again, Case 2.2 is given for A_{k-2}, namely, A_{k-2} consists only of triangles. We can repeat the same arguments inductively.

In the case that k was even, induction leads to $B_0 = \{f_0\}$ being a single triangle. Its three vertices would have to be labeled as $a^+, b, \ldots, a^+, b$, which is not possible for parity reasons. In the case that k was odd, we end up with B_1 consisting of the center face f_0 and triangles attached to each of the edges of f_0. The property $|v|^e_{B_1} = 1$ of each b-vertex of ∂B_1 then implies $|v| = 4$ which yields a contradiction to nonpositive curvature. $\square$

4. Proof of Theorem 3

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be a plane tessellation with nonpositive curvature. We choose $f_0 \in \mathcal{F}$ and define $B_k := B_k(f_0)$ and $A_k = B_k \backslash B_{k-1}$. Note that $B_0 = \{f_0\}$. Let $u \in C_c(\mathcal{V})$ be a compactly supported eigenfunction.

By Lemma 8, B_k is a polygon for every $k \in \mathbb{N}_0$, i.e., the boundary of B_k defines a simple closed path of vertices. In particular, the boundary is a Jordan curve and thus divides the plane in an interior and an exterior part. Therefore, we can define the set of vertices $\mathcal{V}_k$ by

$$\mathcal{V}_k := \{\text{vertices on } \partial B_k\} \cup \{\text{vertices outside of } B_k\}.$$

Theorem 3 follows if we prove the following two steps:

i) There exists a $n_0 \in \mathbb{N}$ such that u vanishes on $\mathcal{V}_{n_0}$.
ii) If u vanishes on $\mathcal{V}_k$ for some $k \in \mathbb{N}$, then u vanishes on $\mathcal{V}_{k-1}$ as well.

Here, i) is immediate from the local finiteness of the graph. Thus, the main point is to prove ii).

To prove ii), we need the following lemma.

Lemma 13. *Let $u \in C_c(\mathcal{V})$ be an eigenfunction of the elliptic operator L on $C(\mathcal{V})$ with $u|_{\mathcal{V}_{k+1}} \equiv 0$. Then, $u(v) = 0$ for all vertices $v \in \partial B_k$ with $|v|^e_{B_k} > 1$.*

Proof. Let $v \in \partial B_k$. If $|v|^e_{B_k} > 2$ then, obviously, v is also a vertex of ∂B_{k+1} and we have $u(v) = 0$. Thus, we only have to consider the case $|v|^e_{B_k} = 2$:

Choose $f \in A_k$ with $v \in \partial f$. Let $f' \in A_{k+1}$ be a neighbour of f. $|v|^e_{B_k} = 2$ implies that f' shares a common edge e, emanating from v, with a neighbour f'' in A_{k+1}. Assume that e connects the vertices v and v'. Since two faces have at most one edge in common, we conclude that v' belongs to ∂B_{k+1}. Now, v' cannot

belong to ∂B_k, for otherwise f' would be in the cut locus $C(f_0)$ (which is empty by Theorem 9). Now, we inspect the other vertices adjacent to v'.

Case 1: All vertices adjacent to v' and different to v belong to $\mathcal{V}_{k+1}$.

Then, none of these vertices belongs to ∂B_k. Thus, we conclude from the fact $u(v') = 0$ that

$$0 = (\lambda - a(v', v'))\, u(v') = a(v', v)\, u(v) + \underbrace{\sum_{v'' \sim v', v'' \neq v} a(v', v'')\, u(v'')}_{=0},$$

and thus $u(v) = 0$ as $a(v', v) \neq 0$ by ellipticity.

Case 2: There is a vertex $w \neq v$ adjacent to v' not belonging to $\mathcal{V}_{k+1}$.

As w is different from v, we infer $|v'|^i_{B_{k+1}} > 2$. Thus, there is at least one face $\hat{f} \in \{f', f''\} \subset A_{k+1}$ which satisfies $v' \in \partial \hat{f}$ and $\partial \hat{f} \cap \partial B_{k+1} = v'$. Therefore, the boundary cycle ∂B_{k+1} does not share an edge with the face $\hat{f} \in A_{k+1}$, and we have $\hat{f} \in C(f_0)$, contradicting to Theorem 9. □

In the following we let

$$v_0, v_1, v_2, \ldots, v_{l-1}, v_l = v_0 \tag{4}$$

be the closed simple sequence of vertices describing ∂B_k.

Lemma 14. *Let $u \in C_c(\mathcal{V})$ be a compactly supported eigenfunction of an elliptic operator on L with $u|_{\mathcal{V}_{k+1}} \equiv 0$. If u vanishes for two subsequent vertices of the sequence* (4)*, then u vanishes on all of ∂B_k.*

Proof. Assume that $u(v_j) = u(v_{j+1}) = 0$. The vertex v_{j+2} satisfies either $|v_{j+2}|^e_{B_k} > 1$, in which case we have $u(v_{j+2}) = 0$, by Lemma 13, and we can continue by considering the subsequent vertices v_{j+1}, v_{j+2}. Otherwise, we have $|v_{j+2}|^e_{B_k} = 1$, in which case v_{j+2} carries label b with respect to the polygon B_k, for curvature reasons.

Since the distance ball B_k is admissible, it follows that v_{j+1} is an a^+-vertex, i.e., we have $|v_{j+1}|^i_{B_k} = 1$. This implies that v_j and v_{j+2} are the only vertices in $\overline{B_k}$ which are adjacent to v_{j+1}. Thus, all other vertices adjacent to v_{j+1} belong to $\mathcal{V}_{k+1}$ and u vanishes on them.

Defining $\mathcal{V}'_{k+1} := \mathcal{V}_{k+1} \backslash \{v_j, v_{j+2}\}$, we can therefore calculate

$$\begin{aligned} 0 &= (\lambda - a(v_{j+1}, v_{j+1}))\, u(v_{j+1}) \\ &= a(v_{j+1}, v_j)\, \underbrace{u(v_j)}_{=0} + a(v_{j+1}, v_{j+2})\, u(v_{j+2}) + \underbrace{\sum_{v \sim v_{j+1}, v \in \mathcal{V}'_{k+1}} a(v_{j+1}, v)\, u(v)}_{=0} \\ &= a(v_{j+1}, v_{j+2}) u(v_{j+1}). \end{aligned}$$

Thus, by ellipticity, we conclude $u(v_{j+2}) = 0$.

Hence, we can also continue with the subsequent vertices v_{j+1}, v_{j+2} in this case. The lemma follows now by iteration. □

Proof of Theorem 3. We follow our strategy and show that $u|_{\mathcal{V}_{k+1}} \equiv 0$ yields $u|_{\partial B_k} \equiv 0$.

By Lemma 13, u vanishes on all vertices with $|v|^e_{B_k} > 1$. If v is an edge which does not satisfy $|v|^e_{B_k} > 1$, it must satisfy $|v|^e_{B_k} = 1$. Then v carries an b-label by Proposition 6. As B_k is admissible it is then enclosed by two a^+-vertices v_{j-1} and $v_{j+1} \pmod{l}$. By Proposition 6 again, these vertices have exterior degree at least 2 and then, by Lemma 13, u vanishes on them.

These considerations show that u vanishes at least for every second vertex of ∂B_k. Now, by Lemma 14, there remains only one case for $u|_{\partial B_k} \not\equiv 0$: l in (4) is

even and, by the admissibility of distance balls, the corresponding label-sequence is $a^+, b, a^+, b, a^+, b, \ldots, a^+, b$, where every a^+-vertex satisfies $|v|^i_{B_k} = 1$ (by definition) and every b-vertex satisfies $|v|^e_{B_k} = 1$. But this case is impossible by Proposition 12. $\square$

5. Proof of Theorem 5

The basic idea in the proof of Theorem 5 is that the existence of a face with a negatively curved corner together with repetitivity implies exponential growth of the number of faces in a combinatorial distance ball. On the other hand, geometrical finiteness implies that combinatorial and Euclidean balls are comparable and that the number of faces inside a Euclidean ball can only grow quadratically with the radius. Obviously, both growth properties are contradictory and $\mathcal{G}$ is forced to have zero curvature.

Assume that $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ is geometrically finite (with generators $f_1, \ldots, f_N \in \mathcal{F}$) and repetitive. Nonpositive curvature implies that we have no cut-locus. We prove that $\kappa(v,f) = 0$ for all corners. Assume, there is $(v,f) \in \mathcal{C}$ with $\kappa(v,f) < 0$. By repetitivity, there is a constant $C > 0$ and a radius $R > 0$, such that every Euclidean ball $B_R(x) \subset \mathbb{R}^2$ contains a face f with

$$\chi(f) := \sum_{v \in \bar{f}} \kappa(v,f) \leq -C.$$

We choose $f_0 \in \mathcal{F}$ arbitrarily and denote $B_k(f_0)$, shortly, by B_k. By geometrical finiteness, there are constants $0 < d < D$ and a point $x_0 \in \mathbb{R}^2$ such that we have, for large enough k:

$$B_{kd}(x_0) \subset \bigcup_{f \in B_k} \bar{f} \subset B_{kD}(x_0).$$

By volume comparison, we immediately obtain the following two facts:

i) There is a constant $c_1 > 0$ such that, for k large enough, $B_{kd}(x_0)$ contains at least $c_1 k^2$ disjoint Euclidean balls of radius R.

ii) There is a constant $c_2 > 0$ such that, for k large enough, $B_{kD}(x_0)$ contains at most $c_2 k^2$ faces of $\mathcal{F}$.

Therefore, the mean Euler-characteristic $\overline{\chi}(B_k) := (\sum_{f \in B_k} \chi(f))/|B_k|$ of distance balls B_k satisfies, for all k large enough,

$$\overline{\chi}(B_k) = \frac{1}{|B_k|} \sum_{f \in B_k} \chi(f) \leq -\frac{c_1}{c_2} C,$$

where $|B_k|$ denotes the number of faces in B_k. By the remark on page 156 of [BP1], this implies that $|B_k(f_0)|$ grows exponentially in k, which contradicts ii).

Consequently, the plane tessellation has zero curvature in all corners, and this immediately yields for each corner $(v,f) \in \mathcal{C}$: $(|v|, E_{\partial f}) \in \{(3,6),(4,4),(6,3)\}$. Finally, face to face extension forces $\mathcal{G}$ to be a regular tiling of type $(3,6),(4,4)$ or $(6,3)$.

6. Further Remarks

In the previous sections we have undertaken some first steps into investigating the geometric situation leading to compactly supported eigenfunctions. We could show that their existence is connected to curvature properties of the underlying graph in the two dimensional situation. This raises various questions:

- Do similar results hold in arbitrary dimension?
- What are sufficient condition for existence of compactly supported eigenfunctions?

- Can one develop a general framework of random operators covering the connection between compactly supported eigenfunctions and the discontinuities of the integrated density of states?

We plan to attack these questions in the future.

References

[Ag] S. Agmon, *Lower bounds for solutions of Schrödinger equations*, J. Analyse Math. **23** (1970). 1–25

[ABG] Amrein, W. O.; Berthier, A.-M.; Georgescu, V., *L^p-inequalities for the Laplacian and unique continuation*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 3, vii, 153–168.

[Aro] N. Aronszajn, *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, J. Math. Pure Appl. **36** (1957), 235-249

[BP1] O. Baues, N. Peyerimhoff, *Curvature and geometry of tessellating plane graphs*, Discrete Comput. Geom. **25** (2001), no. 1, 141-159

[BP2] O. Baues, N. Peyerimhoff, *Geodesics in Non-Positively Curved Plane Tessellations*, preprint, see http://www.math.ethz.ch∼oliver/

[deV] Y. Colin de Verdiere, *Spectres de Graphes*, Soc. Math. de France, No. 4, 1998

[DS] F. Delyon, B. Souillard *Remark on the continuity of the density of states of ergodic finite difference operators*, Comm. Math. Phys. **94** (1984) 289-291

[DLMSchY] J. Dodziuk, P. Linnell, V. Mathai. T. Schick, S. Yates, *Approximating L^2-invariants, and the Atiyah conjecture*, Commun. Pure Appl. Math. **56** (2003), no. 7, 839-873

[GS] B. Grünbaum, G. C. Shepard, *Tilings and patterns*, W. H. Freeman and Company, 1987

[H] L. Hörmander, *Uniqueness theorems for second order elliptic partial differential equations*, Comm. Part. Diff. Equations **8** (1983), 21–64

[JK] D. Jerison, C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. **121** (1985), 463–488

[KLS] S. Klassert, D. Lenz, P. Stollmann, *Discontinuities of the integrated density of states for random operators on Delone sets*, Comm. Math. Phys. **241** (2003) 235-243

[Ves] I. Veselić, *Spectral analysis of percolation Hamiltonians*, Preprint 2004